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Winning Against the Devil: A New Look into the Angel Problem

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HONORS THESIS

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1 Introduction

First introduced by Elwyn R. Berlekamp, John H. Conway and Richard K. Guy in 1982, the angel problem is a combinatorial game played on an infinite playing field, with two players: one angel and one devil. Players alternate turns moving around the board. The angel moves the same way a traditional chess king does on a chess board, one square at a time. The devil can choose any one unoccupied square each turn, ”destroying” the square so the angel can no longer occupy it. The devil wins if it can trap the angel within a destroyed area, and the angel ”wins” if it indefinitely evades capture by the devil.[1]

It was proven by Berlekamp and Conway that the devil has a winning strategy when the angel moves one square at a time. Martin Kutz [2] discusses and analyzes the winning strategy in his 2004 doctoral thesis *The Angel Problem, Positional Games, and Digraph Roots*. This strategy requires the devil to choose a finite size area within the infinite playing field, and then trap the angel within this finite area by blocking the boundary of the area. Variants of the original game were created to find wins for the angel, who could never definitely win on the original infinite playing field. These variants also introduce the notion of giving an angel a ”power”, where the angel can make more than one king move per turn, and ask if a win can be proven for any of the infinite number of powers. Proven wins and strategies were found by András Máthé, Brian Bowditch, and Imre Leader and Béla Bollobás on both finite and infinite playing fields.

2 A New Approach

Following the spirit of these variants of the angel problem, we will define a new angel, named the duck, with a new type of permitted movement.The playing field for our new variant will be the infinite integer Cartesian grid, with the duck starting at the origin (0,0). Let us allow this duck to move only North, East, or diagonally Southwest. The resulting variant is called the duck problem, and is the focus of this research. For our duck problem, there are some preliminary definitions:

Definition: A *duck* moves in three directions, defined by $(i \pm 1, j)$, $(i, j \pm 1)$ and $(i - 1, j - 1)$, (i, j) being the duck's current position.

Definition: A *fox* plays identically to the original devil, blocking any unoccupied square each turn in an attempt to trap the duck.

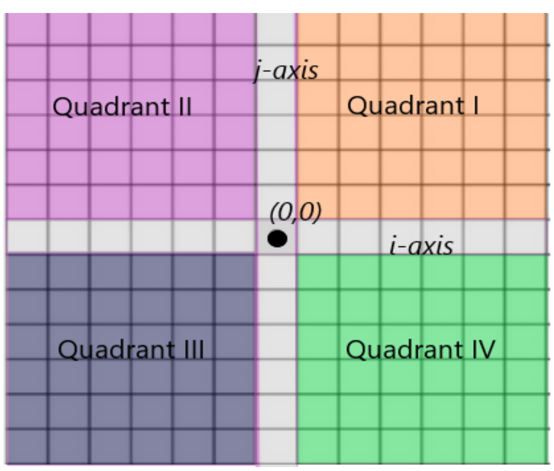

Figure 1: The integer Cartesian grid.

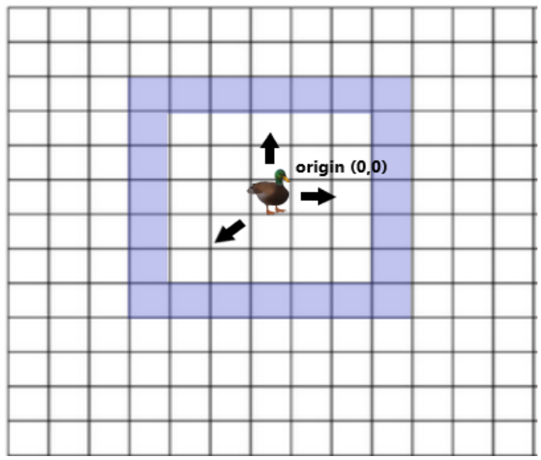

Figure 2: The new duck, starting at origin square (0,0) of the integer Cartesian grid.

Using the same type of strategy laid out by Kutz in his doctoral thesis, the fox-devil will choose a finite size area of a certain quality size S.

Definition: The *quality size* S of the fox's chosen finite area A is defined to be $S = max\{|i|, |j| \mid (i,j) \in A\}$.

First, we must lay out some rules for our duck and fox.

1. The duck can only move one square forward North, right East, or diagonally Southwest.

2. The fox can block any one unoccupied square on the board.

3. The duck will begin at the origin (0,0) of the integer Cartesian grid, and has the first move.

4. The fox wins if it traps the duck-angel on the board, and the duck cannot escape in any permitted direction.

If the duck escapes the chosen finite area, the fox can increase the quality size of the area, and try to trap the duck again within this bigger area. This raises the question ”What is the smallest quality size a fox can choose to trap the duck?” Before attempting to answer this question, the needed strategy must be laid out.

Definition: As the chosen finite area, we define an *S-board* as $B_s = \{(i,j) \mid -S \leq i,j \leq S\}$, with $(2S+1)^2$ total squares. This area is centered at the origin of the integer Cartesian grid.

Definition: A square is a *boundary square* if and only if its coordinates are of the form $(-S, \pm j), (S, \pm j), (\pm i, S)$, or $(\pm i, -S)$.

Definition: A square is a *sub-boundary square* if and only if its coordinates are of the form $(-S+1, \pm j), (S-1, \pm j), (\pm i, S-1)$, or $(\pm i, -S+1), i, j \in \{0, 1, 2 ... S-1\}$.

Definition: The set of all boundary squares is called the *boundary* of the $S-board$.

Definition: The set of all sub-boundary squares is called the *sub-boundary* of the $S-board$.

Definition: The set of all other squares within the $S - board$ but not in the boundary or sub-boundary, is called the *interior* of the $S - board$, defined by $I = \{(i, j) \mid i, j \in (-1, 0, 1)\}$.

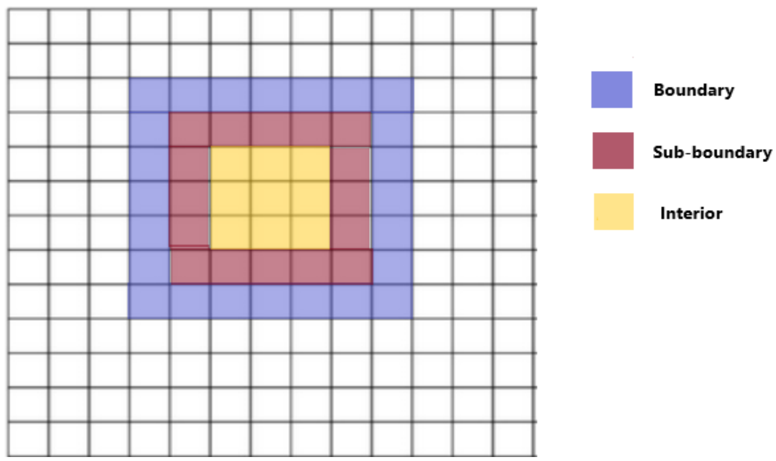

Figure 3: Boundary, sub-boundary, and interior of a 3-board.

Theorem 1: The duck can escape the board if and only if it moves onto a boundary square.

Proof: Suppose the duck is on a boundary square. For any boundary square, the duck can win on its next move by escaping the $S - board$. If it is on the North edge of the boundary, let it move North. If it is on the East edge of the boundary, let it move East. If it is on either the South or West edge of the boundary, let it move Southwest. Since the fox cannot block outside the $S - board$, the duck will escape on its next move. QED.

Theorem 2: The duck starting at the origin (0,0) must pass through the sub-boundary to reach the boundary.

Proof: Let $max\{|i|, |j|\} = S - 1$ for any sub-boundary square with coordinates (i, j) and $max\{|i|, |j|\} = S$ for any boundary square with coordinates (i, j). For the duck's position after the k^{th} move, $P_k = (i_k, j_k)$, define distance: $D(P_k) = max|i|, |j|$. We note the following properties of D:

1. $D(P_0) = 0$ and
2. $|D(P_k) - D(P_{k-1})| \in 0, 1$.

Since $max\{|i|, |j|\} = S$ for any boundary square, the duck must pass $S - 1$, by $|D(P_k) - D(P_{k-1})| \in 0, 1$. Thus, the duck must pass through the sub-boundary to reach the boundary. QED.

Corollary 2.1: The duck loses if and only if it is blocked from moving through the sub-boundary.

Theorem 3: The game will end after at least S moves and at most $(2S+1)^2 - 1$ moves.

Proof: Let the duck start at the origin of the $S - board$. Suppose the duck moves in one direction towards the boundary and the fox does not block its path. The boundary is S squares from the origin, by definition of $S - board$, thus the duck will have reached the boundary in S moves and win, by Theorem 1.

Now, suppose both the duck moves and the fox blocks in a way such that the duck reaches its $(2S+1)^2 - 1th$ move. Let the duck make its $(2S+1)^2th$ move and let the fox do so as well. Since there are only $(2S+1)^2$ squares on any $S - board$, the last unblocked square is the square on which the duck resides on its $(2S+1)^2 - 1th$ move. Since the duck cannot move onto any blocked square, the duck is trapped and cannot make its $(2S+1)^2th$ move. Therefore, the game ends after the $(2S+1)^2 - 1th$ move. QED.

Before proving any wins for the fox with quality size S, there are two cases where the fox-devil does not have a winning strategy. In the case of $S = 1$, a 3x3 grid, for any direction the duck moves on its first turn, it will be in the boundary, and thus, escape the $1 - board$. In the case of $S = 2$, a 5x5 grid, when the duck makes its first move to the North or to the East, on the second move, it will either be on the boundary or on a key square with two unblocked escape routes into the boundary.

Definition: A *key square* is a square in the sub-boundary that has more than one duck-adjacent squares into the boundary.

For any $S > 0$, the key squares are at points $(-S+1, S-1), (S-1, S-1), (S-1, -S+1)$. On the finite area $S = 2$, these key squares are quickly reached within two moves of the duck. Is this the same for $S = 3$, a 7x7 grid?

Using an opening analysis as one would in chess, we can examine all the possible plays within the first two turns of the duck. Let N, E, and D denote a move by the duck to the North, East, and diagonal Southwest, respectively.

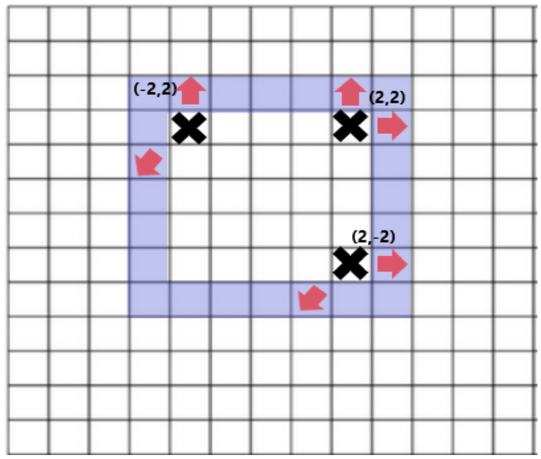

Figure 4: Key squares of a 3-board and the duck-adjacent squares into the boundary.

Suppose the duck makes the following moves and the fox chooses the following squares in response:

NN : The fox will block the closest key square at $(2, 2)$, then the boundary square at $(0, 3)$.

NE : The fox will block the closest key square at $(2, 2)$, then, since the duck will not be on the sub-boundary, the second closest key square at $(2, -2)$.

ND : The fox will block the closest the key square at $(2, 2)$, then, similarly to NE, the second closest key square at $(-2, 2)$.

EN : The fox will block the key square at $(2, 2)$, then the second closest key square at $(2, -2)$.

EE : The fox will block the closest key square at $(2, 2)$, then the boundary square at $(2, 0)$.

ED : The fox will block the closest key square at $(2, 2)$, then the second closest key square at $(2, -2)$.

DN : The fox will block the square at $(-2, -2)$ to prevent a direct escape, then will block the closest key square at $(-2, 2)$.

DE : The fox will block the square at $(-2, -2)$, then the closest key square at $(2, -2)$.

DD : This start for the duck is not possible since the its first move diagonally results in the fox's block on $(-2, 2)$.

What we can observe from this strategy is the prioritization of blocking the key squares within the same Cartesian quadrant the duck currently resides, if it is not on the sub-boundary. When the duck is on the sub-boundary, the fox blocks the boundary square within one move of the duck.

Definition: The *prioritization strategy* is the strategy outlined above.

Definition: A *reactionary strategy* for the fox is as follows:

If the duck is in the sub-boundary:

1. The fox will first move to block any open boundary square that the duck could move to on its next move.

2. If more than one such boundary square is open, the fox will choose randomly (coin-toss). This situation is a losing situation.

3. If no such move is available, the fox blocks the square from which the duck moved.

If the duck is not in the sub-boundary, the fox blocks the square from which the duck moved.
Let the fox utilize both the prioritization and reactionary strategies.

Theorem 4: Once all key squares are blocked, the fox can utilize the reactionary strategy.
Proof: There exist three key squares on any finite area $S > 0$, by definition of key square. Suppose the fox has blocked all three key squares. There exist no more key squares to prioritize. Thus, the fox is free to utilize the reactionary strategy, and the losing situation in 2 cannot occur. QED.

Corollary 4.1: When there no longer exist any key squares, all unblocked sub-boundary squares have exactly one duck-adjacent square into the boundary.

Claim: Combining the reactionary strategy with the prioritization of blocking key squares, the fox will win when $S = 3$.

Proof: Let the duck begin at the origin of the $3 - board$. Let the duck move as described in the opening analysis of the prioritization strategy. By the end of the fox's second turn, at least one key square will be blocked.
Case 1: Suppose the duck has moved in a way that the fox has blocked only one key square. For the four move-combinations that result in only one key square be-

ing blocked, two of these combinations leave the duck on the sub-boundary, and the other two in the interior third quadrant of the board.

Case 1a: Suppose the duck is on the sub-boundary.

By definition of the prioritization strategy, the fox will block the boundary square that is within one move of the duck. The duck will then either move back into the interior or continue along the sub-boundary. If the duck continues along the sub-boundary, let the fox repeat case 1a until the duck reaches a key square, by definition a sub-boundary square, and moves into the interior. If the duck moves into the interior, let the duck then follow case 1b.

Case 1b: Suppose the duck is in the interior. Let the duck block one of the remaining two key squares. Then, if the duck moves into the sub-boundary, let the fox block the boundary square that is within one move of the duck. Else, let the fox block one of the two key squares remaining, then follow case 1a for the next move until the last key square is blocked.

Case 2: Suppose the duck has moved in a way that the fox has blocked two key squares. For all the move-combinations that result in two key squares being blocked after the second move, the duck is left in the interior of the board, and is within at least two moves of the sub-boundary. Thus, after the duck's third move in any direction, it will still be in the interior. So, let the fox block the third key square, by the prioritization strategy. By theorem 4, the fox is free to use the reactionary strategy the rest of the game, and by corollary 4.1, all unblocked sub-boundary squares have exactly one duck-adjacent square into the boundary. Let the duck continue to move on the board and let the fox follow the reactionary strategy until the duck moves into the sub-boundary.

Claim: When the duck is on any remaining sub-boundary square, the duck-adjacent boundary squares can always be blocked by the fox in one move. Proof: Let the duck be on the sub-boundary. Suppose there exists a duck-adjacent boundary square that cannot be blocked by one move of the fox. There are two cases where this could be true:

Case 1: There exists more than one unblocked duck-adjacent boundary square from the sub-boundary square on which the duck resides. Thus by definition, the sub-boundary square is a key square. However, there no longer exist any key squares, and by contradiction, the duck-adjacent square can be blocked.

Case 2: The duck-adjacent boundary square was already blocked on an earlier move. Thus, by contradiction, the duck-adjacent square that cannot be blocked does not exist.

In any case, the remaining duck-adjacent boundary squares can always be blocked, and thus the duck cannot escape through the sub-boundary into the boundary. Therefore, the duck loses by the contrapositive of theorem 1. QED.

3 Conclusions and Future Work

We have now shown that our new angel, the duck, cannot always win on an infinite playing field. Following closely to the original strategy given by Berlekamp in 1982 [2], the fox can choose a finite area within the infinite field and successfully trap the duck within the chosen finite area and thus by extension, any larger finite area. But what would happen if the ”power,” mentioned earlier in this paper, was greater than one? Can a duck of power 2, the ability to move up to two squares in one direction, successfully escape the fox? What about powers 3, 4, or 5? What if the fox also has a power greater than one, the ability to block more than one square at a time? Future works can explore what this notion means for the fox’s strategy. Utilization of computer simulations, Monte Carlo simulations, or ML could help explore the infinite possibilities.

References

[1] Berlekamp, Elwyn R.; Conway, John H.; Guy, Richard K., ”Chapter 19: The King and the Consumer”, Winning Ways for your Mathematical Plays, Volume 2: Games in Particular, Academic Press, 1982, pp. 607–634.

[2] Kutz, Martin. “The Angel Problem, Positional Games, and Digraph Roots Strategies and Complexity.” Berlin, Freie Univ., Diss, 2004, pp. 1–5.